NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

ON THE STABILITY OF THE SOLUTIONS OF EQUATIONS WITH RETARDED ARGUMENTS

By Y. M. Repin

Translation

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E-116

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ABSTRACT

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INDEX HEADING

Research Technique, Mathematics 9.2.7

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E-116

CM-1

In the present paper certain theorems are proved on the stability of the solutions of systems of differential equations with retarded argument. These theorems refer to problems of stability to the first approximation and the behavior of the solutions for small changes of the retardations. They are analogous to the corresponding theorems on stability to the first approximation (ref. 1) and the stability for continuously acting disturbances (ref. 2) for systems of ordinary differential equations and are proven by the methods first applied in the work referred to.

1. Preliminary Remarks

Let us consider the system of equations

$$\frac{dx_i(t)}{dt} = X_i(t,\ x_1(t - \tau_1(t)),\ldots,\ x_1(t - \tau_m(t)),\ldots,\ x_n(t - \tau_1(t)),\ldots,\ x_n(t - \tau_m(t))) \qquad (i = 1,\ldots,\ n)$$

In what follows we shall for briefness write it in the following form:

$$\frac{dx_i}{dt} = X_i(t,\ x_k(t - \tau_j(t))) \quad (i,\ k = 1,\ldots,\ n;\ j = 1,\ldots,\ m) \tag{1.1}$$

The functions $X_i(t,x_{kj})$ depending on $mn + 1$ arguments we shall assume determined and continuous with respect to the set of arguments

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for

$$t \geq A \quad \text{and} \quad |x_{1j}| + \ldots + |x_{nj}| < H$$

It is assumed moreover that $X_i(t,0) \equiv 0$ and that the functions $X_i(t,x_{kj})$ satisfy the Lipshitz condition with respect to the arguments x_{kj} (uniformly with respect to t):

$$\left|X_i(t,x_{kj}) - X_i(t,x_{kj}^o)\right| \leq L \sum_{j=1}^{m} \sum_{k=1}^{n} \left|x_{kj} - x_{kj}^o\right| \quad (t \geq A) \qquad (1.2)$$

The functions $\tau_j(t)$, defined for $t \geq A$, are assumed nonnegative, continuous, and bounded: $\tau_j(t) \leq \tau$.

The fundamental initial problem is formulated as follows. Let there be given $t_0 \geq A$ and a system of n continuous functions $\varphi_i(t)$ on the segment $[t_0 - \tau, t_0]$; it is required to find a system of n continuous functions $x_i(t)$, $t \geq t_0$, satisfying the condition $x_i(t_0) = \varphi_i(t_0)$ and the system (1.1), where if $t - \tau_j(t) \leq t_0$, then $x_k(t - \tau_j(t))$ on the right sides of the system (1.1) must be substituted for $\varphi_k(t - \tau_j(t))$.

As is known (ref. 3), if $|\varphi_1(t)| + \ldots + |\varphi_n(t)| < H$, this problem for the assumptions made has a single solution determined in a certain right-half neighborhood of the point t_0. We shall say that this solution is determined at the instant t_0 by the functions $\varphi_i(t)$. It is evident that the conditions $\varphi_i \equiv 0$ for $A - \tau \leq t \leq A$ determine the trivial solution of the system (1.1) $x_i(t) \equiv 0$, $t \geq A$.

In what follows it is assumed that the solutions determined by the initial functions $\varphi_i(t)$ (at any instant t_0) satisfying the inequality

$$|\varphi_1(t)| + \ldots + |\varphi_n(t)| < \delta < H$$

for sufficiently small δ can be extended over the entire half-axis $t \geq t_0$.

<u>Definition 1</u>. - The trivial solution of the system (1.1) is called uniformly and asymptotically stable if there exists a number $\beta > 0$ such that for each $\eta > 0$ there exists a number $T(\eta) > 0$ such that if

$t - t_0 > T(\eta)$, then

$$|x_1(t)| + \ldots + |x_n(t)| < \eta$$

where $x_i(t)$ are determined from the system (1.1) at the instant t_0 by the functions $\varphi_i(t)$ satisfying the inequality $|\varphi_1(t)| + \ldots + |\varphi_n(t)| < \beta$. Together with the system (1.1), we now consider the system

$$\frac{dx_i^0}{dt} = X_i(t, x_k^0(t - \tau_j^0(t))) \quad (t,k = 1,\ldots, n;\ j = 1,\ldots, m) \qquad (1.3)$$

obtained from system (1.1) by substituting for the functions $\tau_j(t)$ the functions $\tau_j^0(t)$ satisfying the same conditions as $\tau_j(t)$.

<u>Definition 2</u>. - The trivial solution of the system (1.1) is called stable for continuously acting disturbances of the retardations if for any $\varepsilon > 0$ there exist numbers $\delta(\varepsilon)$ and $\rho(\varepsilon)$ such that if

$$|\varphi_1(t)| + \ldots + |\varphi_n(t)| < \delta(\varepsilon), \qquad |\tau_j(t) - \tau_j^0(t)| < \rho(\varepsilon)$$

then

$$|x_1^0(t)| + \ldots + |x_n^0(t)| < \varepsilon \quad \text{for} \quad t \geq t_0$$

where $x_i^0(t)$ are determined from the system (1.3) by the initial functions $\varphi_i(t)$ at the instant t_0.

With regard to definition 2, see references 4 and 5. We shall prove two auxiliary propositions.

<u>Lemma 1</u>. - For $t_0 \leq t \leq T$, let a continuous function satisfy the inequality

$$M(t) \leq f(t) + C \int_{t_0}^{t} M(\xi)\, d\xi$$

where $f(t)$ is continuous on $\left[t_0, T\right]$. Then,

$$M(t) \leq f(t) + C \int_{t_0}^{t} f(\xi)\, e^{c(t-\xi)}\, d\xi \qquad (1.4)$$

In particular, if $f(t)$ has a continuous derivative, then

$$M(t) \leq f(t)\, e^{c(t-t_0)} + \int_{t_0}^{t} f'(\xi)\, e^{c(t-\xi)}\, d\xi \qquad (1.5)$$

<u>Proof</u>. - It is not difficult to see that

$$M^o(t) = f(t) + C \int_{t_0}^{t} f(\xi)\, e^{c(t-\xi)}\, d\xi$$

is a solution of the integral equation

$$M^o(f) = f(t) + C \int_{t_0}^{t} M^o(\xi)\, d\xi$$

To prove inequality (1.4), it is sufficient to establish that

$$N(t) = M^o(t) - M(t) \geq 0$$

But, obviously,

$$N(t) \geq C \int_{t_0}^{t} N(\xi)\, d\xi$$

Since

$$N(\xi) \geq C \int_{t}^{\xi} N(\eta)\, d\eta$$

we can write

$$N(t) \geq C^2 \int_{t_0}^{t} \int_{t_0}^{\xi} N(\eta)\, d\eta\, d\xi = C^2 \int_{t_0}^{t} (t - \xi)\, N(\xi)\, d\xi$$

Continuing to substitute under the integral $C\int_{t_0}^{\xi} N(\eta)\,d\eta$ for $N(\xi)$, we obtain

$$N(t) \geq \frac{c^n}{(n-1)!}\int_{t_0}^{t} (t-\xi)^{n-1} N(\xi)\,d\xi \qquad (n = 1,2,\ldots)$$

Passing, in this inequality, to the limit as $n \to \infty$, we obtain $N(t) \geq 0$, as was required. Inequality (1.5) is obtained from this by integrating by parts.

Lemma 2. - The solution of system (1.1), determined at the instant $t_0 \geq A$ by the functions $\varphi_i(t)$ satisfying the inequality $|\varphi_1(t)| + \ldots + |\varphi_n(t)| < \delta < H$, satisfies the inequality

$$\sum_{i=1}^{n} |x_i(t)| \leq \delta e^{mnL(t-t_0)} \qquad (1.6)$$

provided it can be continued in the region $|x_1| + \ldots + |x_n| < H$.

Proof. - In virtue of the system (1.1) and the initial conditions, we have

$$x_i(t) = \varphi_i(t_0) + \int_{t_0}^{t} X_i(\xi,\, x_k(\xi - \tau_j(\xi)))\,d\xi$$

whence

$$\sum_{i=1}^{n} |x_i(t)| \leq \delta + nL\int_{t_0}^{t} \sum_{j=1}^{m}\sum_{k=1}^{n} |x_k(\xi - \tau_j(\xi))|\,d\xi$$

where, if $\xi - \tau_j(\xi) \leq t_0$, then $x_k(\xi - \tau_j(\xi)) = \varphi_k(\xi - \tau_j(\xi)$.

Let $M(t) = \max\left\{\delta,\ \max\left[|x_1(\xi)| + \ldots + |x_n(\xi)|\right]\right\}$ for $t_0 \leq \xi \leq t$. Evidently, we have

$$\sum_{i=1}^{n} |x_i(t)| \leq \delta + mnL\int_{t_0}^{t} M(\xi)\,d\xi$$

Since the right side of this inequality monotonically does not decrease and is always not less than δ, the functions $M(\xi)$ satisfy the inequality

$$M(t) \leq \delta + mnL \int_{t_0}^{t} M(\xi)\, d\xi$$

whence according to lemma 1 the required inequality also follows.

E-116

2. Stability in the First Approximation

Together with the system (1.1), let us consider the system

$$\frac{dy_i}{dt} = Y_i(t, y_k(t - \tau_j(t))) \quad (i, k = 1,\ldots, n; \; j = 1,\ldots, m) \quad (2.1)$$

satisfying the same conditions as system (1.1) (the Liptshitz constant for the system (2.1) may, of course, be different).

Theorem 1. - Let the trivial solution of system (1.1) be uniformly and asymptotically stable, where there exist constants $\alpha > 0$, $B \geq 1$ such that for all sufficiently small δ there follows from the inequality $|\varphi_1(t)| +\ldots+ |\varphi_n(t)| < \delta$ the inequality

$$|x_1(t)| +\ldots+ |x_n(t)| < B\delta e^{-\alpha(t-t_0)} \quad (2.2)$$

where $x_i(t)$ are determined from the system (1.1) by the functions $\varphi_i(t)$ at the instant t_0. Further, let the right sides of systems (1.1) and (2.1) satisfy the inequality

$$|X_i(t,x_{kj}) - Y_i(t,x_{kj})| \leq \sigma \sum_{j=1}^{m} \sum_{k=1}^{n} |x_{kj}| \quad \text{for} \quad \sum_{k=1}^{n} |x_{kj}| < h < H \quad (2.3)$$

Then for sufficiently small σ the trivial solution of the system (2.1) is uniformly and asymptotically stable.

Proof. - Let us consider at the instant t_0 a certain system of the initial functions $\varphi_i(t)$ satisfying the condition $|\varphi_1(t)| +\ldots+ |\varphi_n(t)| < \delta$, where $\delta < h$, and so small that the solution of the system (1.1) determined by the functions $\varphi_i(t)$ can be continued on the entire half-axis $t \geq t_0$ and satisfies inequality (2.2). With the same initial functions we determine also the solution of the system (2.1) in a certain right-half neighborhood of the point t_0. We shall show that, provided $|y_1(t)| + \ldots + |y_n(t)| < h$, the inequality will hold:

$$\sum_{i=1}^{n} |x_i(t) - y_i(t)| \leq \frac{\sigma\delta B}{L + \sigma} \left(e^{mn(L+\sigma)(t-t_0)} - 1\right) \tag{2.4}$$

E-116

In fact, on account of (1.1) and (2.1), we have

$$x_i(t) = \varphi_i(t_0) + \int_{t_0}^{t} X_i(\xi, x_k(\xi - \tau_j(\xi)))\, d\xi$$

$$y_i(t) = \varphi_i(t_0) + \int_{t_0}^{t} Y_i(\xi, y_k(\xi - \tau_j(\xi)))\, d\xi$$

Then

$$\sum_{i=1}^{n} |x_i(t) - y_i(t)| \leq \sum_{i=1}^{n} \int_{t_0}^{t} |X_i(\xi, x_k(\xi - \tau_j(\xi))) - Y_i(\xi, y_k(\xi - \tau_j(\xi)))|\, d\xi \leq$$

$$\sum_{i=1}^{n} \int_{t_0}^{t} |X_i(\xi, x_k(\xi - \tau_j(\xi))) - X_i(\xi, y_k(\xi - \tau_j(\xi)))|\, d\xi +$$

$$\sum_{i=1}^{n} \int_{t_0}^{t} |X_i(\xi, y_k(\xi - \tau_j(\xi))) - Y_i(\xi, y_k(\xi - \tau_j(\xi)))|\, d\xi \leq$$

$$n\left[L \int_{t_0}^{t} \sum_{j=1}^{m} \sum_{k=1}^{n} |x_k(\xi - \tau_j(\xi)) - y_k(\xi - \tau_j(\xi))| + \sigma \int_{t_0}^{t} \sum_{j=1}^{m} \sum_{k=1}^{n} |y_k(\xi - \tau_j(\xi))|\, d\xi\right] \leq$$

$$n \int_{t_0}^{t} \left[(L + \sigma) \sum_{j=1}^{m} \sum_{k=1}^{n} |x_k(\xi - \tau_j(\xi)) - y_k(\xi - \tau_j(\xi))| + \sigma \sum_{j=1}^{m} \sum_{k=1}^{n} |x_k(\xi - \tau_j(\xi))|\right] d\xi$$

Let

$$M(t) = \max \sum_{i=1}^{n} |x_i(\xi) - y_i(\xi)| \quad \text{for} \quad t_0 \leq \xi \leq t$$

Then

$$\sum_{i=1}^{n} |x_i(t) - y_i(t)| \leq \sigma mnB\delta(t - t_0) + mn(L + \sigma) \int_{t_0}^{t} M(\xi)\, d\xi$$

whence

$$M(t) \leq mn\sigma\delta B(t - t_0) + mn(L + \sigma) \int_{t}^{t} M(\xi)\, d\xi$$

Applying lemma 1, we obtain the required inequality.

Let us now take an arbitrary number $\varepsilon < h$ so small that if

$$|\varphi_1(t)| + \ldots + |\varphi_n(t)| < \varepsilon/2B$$

then to the difference of the solutions of systems (1.1) and (2.1), determined by the functions $\varphi_i(t)$, the estimate just obtained can be applied.

Let $|\varphi_1(t)| + \ldots + |\varphi_n(t)| < \varepsilon/2B$, and the magnitude $T = (1/\alpha) \ln 4B$ and σ so small that

$$\frac{\sigma B}{L + \sigma} \left(e^{mn(L+\sigma)(T+\tau)} - 1\right) < \frac{1}{4}$$

We shall show that the solution $y_i(t)$ of the system (2.1) determined at the instant t_0 by the initial functions $\varphi_i(t)$, for $t_0 \leq t \leq t_0 + T + \tau$ cannot go outside the ε-neighborhood of the origin of coordinates (here and in the following by ε-neighborhood of the origin of coordinates there is meant the set of points satisfying the inequality $|x_1| + \ldots + |x_n| < \varepsilon$), and for $t_0 + T \leq t \leq t_0 + T + \tau$ lies in the $\varepsilon/4B$-neighborhood of the origin. Whence, in particular, it follows that the solution $y_i(t)$ can be continued also on the segment $[t_0, t_0 + T + \tau]$.

In fact, at the instant $t_0 \leq t_1 \leq t_0 + T + \tau$ let the solution $y_i(t)$ at first go outside the ε-neighborhood of the origin. Then, if $x_i(t)$ is determined from the system (1.1) by the same initial functions as $y_i(t)$,

$$\varepsilon = \sum_{i=1}^{n} |y_i(t_1)| \leq \sum_{i=1}^{n} |x_i(t_1)| + \sum_{i=1}^{n} |x_i(t_1) - y_i(t_1)| < \frac{\varepsilon}{2B} +$$

$$\frac{\varepsilon \sigma B}{2B(L + \sigma)} \left(e^{mn(L+\sigma)(t_1 - t_0)} - 1\right) < \frac{\varepsilon}{-} + \frac{\varepsilon}{8B} < \varepsilon$$

But this is a contradiction. Now for $t_0 + T \leq t \leq t_0 + T + \tau$,

$$\sum_{i=1}^{n} y_i(t) < \frac{\varepsilon}{2} e^{-\alpha T} + \frac{\varepsilon}{8B} = \frac{\varepsilon}{4B}$$

If the $y_i(t)$ are now considered on the segment $[t_0 + T,\ t_0 + T + \tau]$ as the initial functions at the instant $t_0 + T + \tau$, then relative to the continuation of the solution $y_i(t)$ we arrive by exactly the same method at the conclusion that for $t_0 + T + \tau \leq t \leq t_0 + 2(T + \tau)$ it does not go outside the $\varepsilon/2$-neighborhood of the origin, while for $t_0 + 2T + \tau \leq t \leq t_0 + 2(T + \tau)$ it lies in the $\varepsilon/8B$-neighborhood of the origin.

Continuing this process by successive steps of length $T + \tau$, we arrive at the conclusion that, if $\varphi_i(t)$ lies in the $\varepsilon/2B$-neighborhood of the origin, then the solution of the system (2.1) on the segment $[t_0 + n(T + \tau),\ t_0 + (n + 1)(T + \tau)]$ lies in the $\varepsilon/2^n$-neighborhood of the origin. It is now evident that the trivial solution of the system (2.1) is uniformly and asymptotically stable.

<u>Corollary 1</u>. - If the system (2.1) satisfies the condition

$$|X_i(t,x_{kj}) - Y_i(t,x_{kj})| \leq \sum_{j=1}^{m} \sum_{k=1}^{n} |x_{kj}|\ \psi\left(\sum_{j=1}^{m} \sum_{k=1}^{n} |x_{kj}|\right) \text{ for } \sum_{k=1}^{n} |x_{kj}| < H$$

where $\psi(x) \to 0$ for $x = 0$, then under the same assumptions with regard to the system (1.1), the trivial solution of the system (2.1) is uniformly and aymptotically stable. In particular, the system (1.1) may be the system obtained as a result of the usual linearization of the system (2.1).

In fact, for any $\sigma > 0$ the inequality (2.3) is satisfied for sufficiently small h.

Corollary 2. - For the functions $Y_i(t,x_{kj})$, let there exist functions $X_i(x_{kj})$ such that for any $\sigma > 0$ there exists a number $A(\sigma) \geq A$ such that for $t \geq A(\sigma)$ the inequality is satisfied

$$|Y_i(t,x_{kj}) - X_i(x_{kj})| < \sigma \sum_{j=1}^{m} \sum_{k=1}^{n} |x_{kj}|$$

If the solutions of the system

$$\frac{dx_i}{dt} = X_i(x_k(t - \tau_j(t))) \tag{2.5}$$

satisfy the conditions imposed in theorem 1 on the solutions of system (1.1), then the trivial solution of system (2.1) is uniformly and asymptotically stable.

In fact, let us choose σ, as in the proof of theorem 1, and then consider the systems (2.1) and (2.5) for $t \geq A(\sigma)$. Applying theorem 1 and lemma 2 we obtained the proof required.

Remark. - Theorem 1 differs from the theorem of Wright (refs. 4 and 6) on the stability by the first approximation, in particular by the circumstance that the nonstationary case is also taken into account. However, strictly speaking, it is not a generalization of Wright's theorem at least for the reason that in the latter there are considered not only equations with retarded argument but also equations of the 'neutral type' (ref. 4).

3. Maintenance of Stability for Small Changes of Retardations

We consider the systems (1.1) and (1.3).

Theorem 2. - If the solutions of system (1.1) satisfy the conditions imposed on them in theorem 1, and $|\tau_j(t) - \tau_j^0(t)| < \rho$, then for sufficiently small ρ the trivial solution of the system (1.3) is uniformly and asymptotically stable.

E-116

<u>Proof</u>. - Let us assume that the functions $\varphi_i(t)$ determining the solutions $x_i(t)$ and $x_i^0(t)$ of the systems (1.1) and (1.3) at the instant t_0 possess continuous derivatives and let

$$|\varphi_1(t)| + \ldots + |\varphi_n(t)| < \delta, \ |\varphi_1(t)| < \gamma$$

Let δ be so small that the solution $x_i(t)$ of the system (1.1) can be continued on the entire semiaxis $t \geq t_0$ and satisfies the inequality $|x_1(t)| + \ldots + |x_n(t)| \leq B\delta e^{-\alpha(t-t_0)}$.

We shall show that the inequality holds:

$$\sum_{i=1}^{n} \left|x_i(t) - x_i^0(t)\right| \leq \rho n(\gamma + \delta mLB)(e^{mnL(t-t_0)} - 1) \qquad (3.1)$$

provided the solution $x_i^0(t)$ can be continued in the H-neighborhood of the origin. By virtue of (1.1) and (1.3), we have

$$x_i(t) = \varphi_i(t_0) + \int_0^t X_i(\xi, x_k(\xi - \tau_j(\xi)))\, d\xi$$

$$x_i^0(t) = \varphi_i(t_0) + \int_{t_0}^t X_i(\xi, x_k^0(\xi - \tau_j^0(\xi)))\, d\xi$$

Then

$$\sum_{i=1}^{n} |x_i(t) - x_i^0(t)| \leq \sum_{i=1}^{n} \int_{t_0}^{t} |X_i(\xi, x_k(\xi - \tau_j(\xi))) - X_i(\xi, x_k^0(\xi - \tau_j^0(\xi)))|\, d\xi \leq$$

$$nL \int_{t_0}^{t} \sum_{j=1}^{m} \sum_{k=1}^{n} |x_k(\xi - \tau_j(\xi)) - x_k^0(\xi - \tau_j^0(\xi))|\, d\xi \leq$$

$$nL \int_{t_0}^{t} \sum_{j=1}^{m} \sum_{k=1}^{n} |x_k(\xi - \tau_j(\xi)) - x_k(\xi - \tau_j^0(\xi))|\, d\xi +$$

$$nL \int_{t_0}^{t} \sum_{j=1}^{m} \sum_{k=1}^{n} |x_k(\xi - \tau_j^0(\xi)) - x_k^0(\xi - \tau_j^0(\xi))|\, d\xi$$

We now estimate separately

$$\int_{t_0}^{t} \sum_{k=1}^{n} |x_k(\xi - \tau_j(\xi)) - x_k(\xi - \tau_j^0(\xi))| \, d\xi$$

$$= (t - t_0) \sum_{k=1}^{n} |x_k(\theta - \tau_j(\theta)) - x_k(\theta - \tau_j^0(\theta))| \quad (t_0 \le \theta \le t)$$

Here three cases may present themselves. The first case, if

$$\theta - \tau_j(\theta) \le t_0, \quad \theta - \tau_j^0(\theta) \le t_0$$

where

$$\sum_{k=1}^{n} |x_k(\theta - \tau_j(\theta)) - x_k(\theta - \tau_j^0(\theta))| \le \rho\gamma n$$

since $x_k(t)$ for $t \le t_0$ coincides with $\varphi_k(t)$.

The second case occurs if

$$\theta - \tau_j(\theta) \ge t_0 \text{ and } \theta - \tau_j^0(\theta) \ge t_0$$

where

$$\sum_{k=1}^{n} |x_k(\theta - \tau_j(\theta)) - x_k(\theta - \tau_j^0(\theta))| = \sum_{k=1}^{n} \left| \int_{\theta-\tau_j^0(\theta)}^{\theta-\tau_j(\theta)} X_k(\eta, x_\nu(\eta - \tau_\mu(\eta))) \, d\eta \right| \le$$

$$nL \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} \left| \int_{\theta-\tau_j^0(\theta)}^{\theta-\tau_j(\theta)} |x_\nu(\eta - \tau_\mu(\eta))| \, d\eta \right| \le \rho\delta mnLB$$

The third case, if one of the numbers $\theta - \tau_j(\theta)$, $\theta - \tau_j^0(\theta)$ is greater than t_0, but the second less than t_0. We then write the

E-116

inequality

$$\sum_{k=1}^{n} |x_k(\theta - \tau_j(\theta)) - x_k(\theta - \tau_j^o(\theta)| \leq$$

$$\sum_{k=1}^{n} |x_k(\theta - \tau_j(\theta)) - x_k(t_0)| + \sum_{k=1}^{n} |x_k(t_0) - x_k(\theta - \tau_j^o(\theta))|$$

and estimate each of the sums on the right side of this inequality, in both the first and second cases; we obtain

$$\sum_{k=1}^{n} |x_k(\theta - \tau_j(\theta)) - x_k(\theta - \tau_j^o(\theta))| \leq \rho n(\gamma + \delta mLB)$$

It is evident that the last inequality holds for any case. Now let

$$M(t) = \max \sum_{k=1}^{n} |x_k(\xi) - x_k^o(\xi)| \quad \text{for} \quad t_0 \leq \xi \leq t$$

We then obtain

$$\sum_{t=1}^{n} |x_i(t) - x_i^o(t)| \leq \rho mn^2(\gamma + \delta mLB)\ L(t - t_0) + mnL \int_{t_0}^{t} M(\xi)\ d\xi$$

whence

$$M(t) \leq \rho mn^2(\gamma + \delta mLB)\ L(t - t_0) + mnL \int_{t_0}^{t} M(\xi)\ d\xi$$

and applying lemma 1, we obtain the inequality (3.1).

Let us take $\varepsilon > 0$ so small that the inequality (3.1) can be applied if

$$|\varphi_1(t)| + \ldots + |\varphi_n(t)| < \varepsilon/2B$$

Let

$$\sum_{i=1}^{n} |\varphi_i(t)| < \frac{\varepsilon}{2Be^{mnL\tau}}, \; t_0 \geq A, \; T = \frac{\ln 4B}{\alpha}, \; \rho$$

be so small that $\rho mnL(1 + B)(e^{mnL(T+2\tau)} - 1) < 1/4$.

We shall show that on the segment $[t_0 + \tau, t_0 + T + 3\tau]$ the solution $x_i^0(t)$ of the system (1.3), determined by the functions $\varphi_i(t)$ at the instant t_0, cannot go out of the ε-neighborhood of the origin, but on the segment $[t_0 + T + \tau, t_0 + T + 3\tau]$ lies in the $\varepsilon/4B$-neighborhood of the origin. In fact, on the segment $[t_0, t_0 + \tau]$ the inequality holds:

$$|x_1^0(t)| + \ldots + |x_n^0(t)| < \varepsilon/2B$$

in virtue of inequality (1.6).

We shall now determine the solution $x_i(t)$ of the system (1.1) with the functions $x_i^0(t)$ at the instant $t_0 + \tau$. Evidently, $x_i^0(t)$ has continuous derivatives on the segment $[t_0, t_0 + \tau]$, where

$$dx_i^0/dt = X_i(t, x_k^0(t - \tau_j^0(t)))$$

From the last equation it follows that

$$\left|\frac{dx_i^0}{dt}\right| \leq L \sum_{j=1}^{m}\sum_{k=1}^{n} |x_k^0(t - \tau_j^0(t))| \leq \frac{\varepsilon mL}{2B}$$

On the segment $[t_0 + \tau, t_0 + T + 3\tau]$, we now have everywhere the inequality

$$\sum_{i=1}^{n} |x_i^0(t)| \leq \sum_{i=1}^{n} |x_i(t)| + \sum_{i=1}^{n} |x_i(t) - x_i^0(t)| \leq \frac{\varepsilon}{2} e^{-\alpha(t-t_0-\tau)} +$$

$$\rho n \left(\frac{\varepsilon mL}{2B} + \frac{\varepsilon mL}{2}\right)(e^{mnL(t-t_0-\tau)} - 1) \leq \frac{\varepsilon}{2} e^{-\alpha(t-t_0-\tau)} + \frac{\varepsilon}{8B}$$

E-116

The first component everywhere on the segment considered does not exceed $\varepsilon/2$; but if $t_0 + T + \tau \leq t_0 + T + 3\tau$, then it does not exceed $\frac{1}{2}\varepsilon e^{-\alpha T} = \varepsilon/8B$. Whence follows our assertion on the behavior of $x_i^0(t)$.

Let us now consider the solution of the system (1.1) determined by the functions $x_i^0(t)$ at the instant $t_0 + T + 3\tau$. Again, $x_i^0(t)$ has continuous derivatives, where, since $|x_1^0(t)| + \ldots + |x_n^0(t)| < \varepsilon/4B$ on the segment $[t_0 + T + \tau,\ t_0 + T + 3\tau]$,

$$|dx_i^0/dt| < \varepsilon mL/4B$$

Repeating further these steps (of time interval $T + 2\tau$), we arrive at the conclusion that on the segment

$$[t_0 + nT + (2n + 1)\tau,\ t_0 + (n + 1)T + (2n + 3)\tau]$$

the inequality holds:

$$\sum_{i=1}^{n} |x_i^0(t)| < \frac{\varepsilon}{2^n}, \text{ if } \sum_{i=1}^{n} |\varphi_i(t)| < \frac{\varepsilon}{2Be^{mnL}}$$

Whence follows the conclusion of the theorem on the uniform asymptotic stability of the trivial solution of system (1.3).

<u>Corollary</u>. - In the system (1.1), let $\tau_j(t) \to \tau_j$ for $t \to \infty$. If the trivial solution of the system

$$\frac{dx_i}{dt} = X_i(t, x_k(t - \tau_j)) \tag{3.2}$$

is uniformly and asymptotically stable, and inequality (2.2) holds, the trivial solution of the system (1.1) is uniformly and asymptotically stable. In fact, let us choose ρ as in the proof of theorem 2 and let $|\tau_j(t) - \tau_j| < \rho$ for $t \geq A(\rho) \geq A$ for all j. Considering the systems (1.1) and (3.2) for $t \geq A(\rho)$ and applying theorem 2 and lemma 2, we obtain what was required.

E-116

4. Stability for Continuously Acting Disturbances of Retardations

Theorem 3. - If the trivial solution of system (1.1) is uniformly and asymptotically stable, it is stable for continuously acting disturbances of the retardations.

Proof. - Let us consider the system (1.3) together with the system (1.1) and estimate the difference between their solutions, determined at the instant t_0 by the same functions $\varphi_i(t)$, admitting continuous derivatives, with

$$|\varphi_1(t)| + \ldots + |\varphi_n(t)| < \delta < H, \quad |d\varphi_i(t)/dt| < \gamma$$

We shall show that under these assumptions the inequality holds:

$$\sum_{i=1}^{n} |x_i(t) - x_i^0(t)| \le \rho\gamma n(e^{mnL(t-t_0)} - 1) +$$

$$\rho\delta m^2n^2L^2(t - t_0)\left(1 + \frac{mnL(t - t_0)}{2}\right) e^{mnL(t-t_0)} \qquad (4.1)$$

provided both solutions can be continued in the H-neighborhood of the origin. The estimate is conducted exactly in the same way as in section 3, excluding only the second case arising in the estimate

$$\int_{t_0}^{t} \sum_{k=1}^{n} |x_k(\xi - \tau_j(\xi)) - x_k(\xi - \tau_j^0(\xi))| \, d\xi$$

We here have the inequality (1.6):

$$\sum_{k=1}^{n} |x_k(\theta - \tau_j(\theta)) - x_k(\theta - \tau_j^0(\theta))| \le$$

$$nL\sum_{\mu=1}^{m}\sum_{\nu=1}^{n} \left|\int_{\theta-\tau_j^0(\theta)}^{\theta-\tau_j(\theta)}\right| x_\nu(\eta - \tau_\mu(\eta))| \, d\eta \le \rho\delta mnLe^{mnL(t-t_0)}$$

E-116

E-116

Thus,

$$\int_{t_0}^{t}\sum_{k=1}^{n}|x_k(\xi - \tau_j(\xi)) - x_k(\xi - \tau_j^0(\xi))|\, d\xi \le (t - t_0)\, \rho n(\gamma + \delta mLe^{mnL(t-t_0)})$$

and finally

$$M(t) < L\rho mn^2(\gamma + \delta mLe^{mnL(t-t_0)})(t - t_0) + mnL\int_{t_0}^{t} M(\xi)\, d\xi$$

From this we obtain the inequality (4.1) by the application of lemma 1.

We shall now prove that for each $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that, if the initial functions $\varphi_1(t)$ satisfy the condition $|\varphi_1(t)| +\ldots+ |\varphi_n(t)| < \delta$, then the solution of the system (1.1) determined by them will for all $t \geq t_0$ satisfy the inequality $|x_1(t)| +\ldots+ |x_n(t)| < \varepsilon$. In fact, for ε there exists a number $T(\varepsilon)$ such that for $t \geq t_0 + T(\varepsilon)$ we shall have $|x_1(t)| +\ldots+ |x_n(t)| < \varepsilon$ (if $\delta < \beta$).

Let $\delta < \varepsilon/e^{mnLT(\varepsilon)}$. Then also for $t_0 \le t \le T(\varepsilon) + t_0$,

$$|x_1(t)| +\ldots+ |x_n(t)| < \varepsilon$$

in virtue of inequality (1.6).

We now take $\varepsilon > 0$ and corresponding to the number $\varepsilon/2$ we find $\delta_1 > 0$ such that if

$$|\varphi_1(t)| +\ldots+ |\varphi_n(t)| < \delta_1, \text{ then } |x_1(t)| +\ldots+ |x_n(t)| < \varepsilon/2$$

Further, corresponding to the number $\delta_1/2$ we find T such that if $t - t_0 > T$, then

$$|x_1(t)| +\ldots+ |x_n(t)| < \delta_1/2$$

We shall show that if

$$|\varphi_1(t)| +\ldots+ |\varphi_n(t)| < \delta_1 e^{-mnL\tau}$$

and ρ is so small that

$$\rho mLn \left\{ \left(e^{mnL(T+2\tau)} - 1\right) + mnL(T + 2\tau)\left(1 + \frac{mnL(T + 2\tau)}{2}\right) e^{mnL(T+2\tau)} \right\} < \frac{1}{2}$$

then the solution $x_i^0(t)$ of the system (1.3), determined by the functions $\varphi_i(t)$, satisfies the inequality $|x_1^0(t)| + \ldots + |x_n^0(t)| < \varepsilon$ for $t \geq t_0$.

In fact, $|x_1^0(t)| + \ldots + |x_n^0(t)| < \delta_1$ on the segment $[t_0, t_0 + \tau]$. We shall show that

$$|x_1^0(t)| + \ldots + |x_n^0(t)| < \varepsilon \quad \text{on the segment } [t_0 + \tau, t_0 + T + 3\tau]$$

$$|x_1^0(t)| + \ldots + |x_n^0(t)| < \delta_1 \quad \text{on the segment } [t_0 + T + \tau, t_0 + T + 3\tau]$$

In fact, taking as the initial functions for the system (1.1) the functions $x_i^0(t)$ on the segment $[t_0, t_0 + \tau]$, we determine by their means the solution of this system. Applying the inequality (4.1) with $\gamma = \delta_1 mL$ (which is obtained as in section 3), we have

$$\sum_{i=1}^{n} |x_i^0(t)| \leq \sum_{i=1}^{n} |x_i(t)| + \sum_{i=1}^{n} |x_i(t) - x_i^0(t)| < \frac{\varepsilon}{2} +$$

$$\rho\delta_1 mnL \left\{ \left(e^{mnL(t-t_0-\tau)} - 1\right) + \right.$$

$$\left. mnL(t - t_0 - \tau)\left(1 + \frac{mnL(t - t_0 - \tau)}{2}\right) e^{mnL(t-t_0-\tau)} \right\} \leq \frac{\varepsilon}{2} + \frac{\delta_1}{2} < \varepsilon$$

On the segment $[t_0 + T + \tau, t_0 + T + 3\tau]$, however, by the choice of T we have

$$\sum_{i=1}^{n} |x_i^0(t)| < \frac{\delta_1}{2} + \frac{\delta_1}{2} = \delta_1$$

We again consider the solution of system (1.1), determined by the functions $x_i^0(t)$, taken on the segment $[t_0 + T + 2\tau, t_0 + T + 3\tau]$.

By the same considerations we find that $|x_1^0(t)| + \ldots + |x_n^0(t)| < \varepsilon$ also on the segment $[t_0 + T + 3\tau, t_0 + 2T + 5\tau]$, while on the segment $[t_0 + 2T + 3\tau, t_0 + 2T + 5\tau]$ we shall have $|x_1^0(t)| + \ldots + |x_n^0(t)| < \delta_1$.

Continuing this process further we find that

$$|x_1^0(t)| + \ldots + |x_n^0(t)| < \varepsilon \quad \text{for} \quad t \geq t_0$$

as was required. The theorem proved is a generalization of the local part of the theorem of reference 5 on the stability for continuously acting disturbances of the retardations.

E-116

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National Aeronautics and
Space Administration